Linear Form of the Radiative Transfer Equation Revisited

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Outline

• Smith’s linear form *(Applied Optics, 1991)* vs. its dual form.

• The linear form with *exact* analytical Jacobians vs. the tangent-linear and adjoint models.

• The linear form with *exact* analytical Jacobians vs. the linear form with *inexact* analytical Jacobians.

• Why use the forward model with exact analytical Jacobians for physical retrieval and data assimilation?

• Summary

• Future work
Bill Smith’s Achievement in Physical Retrieval

• Bill has been pioneering the physical retrieval of temperature and absorbing constituent profiles from the radiance spectra since the late ’60s.

• His most recent linear form of the RTE was published in the landmark paper in *Applied Optics* (1991).

• This *monochromatically approximate* linear form and its variant have been still used in the physical retrieval [e.g. Ma *et al*., 2000, Li *et al*., 2001].

• His work inspired Huang *et al*. (*Applied Optics*, 2002) to successfully derive the linear form with *exact* analytical Jacobians for the widely-used McMillin-Fleming-Eyre-Woolf type of forward models (e.g. RTTOV, RTIASI).

• In this talk I will prove that there exists the *dual* representation of Smith’s linear form (1991), and show some mathematically interesting outcomes derived from this dual form.
Smith’s Linear Form (1991) vs. its Dual Form

\[ R^\text{obs}_v = B_v(p_s) \tau_v(p_s) - \int_0^{p_s} B_v(p) d\tau_v(p) \]

\[ R^0_v = B^0_v(p_s) \tau^0_v(p_s) - \int_0^{p_s} B^0_v(p) d\tau^0_v(p) \]

\[ \delta R_v \equiv R^\text{obs}_v - R^0_v \]

\[ \delta B_v(p) \equiv B_v(p) - B^0_v(p) \]

\[ \delta \tau_v(p) \equiv \tau_v(p) - \tau^0_v(p) \]

Smith’s Linear Form

\[ \delta R_v = B_v(p_s) \delta \tau_v(p_s) + \delta B_v(p_s) \tau^0_v(p_s) - \int_0^{p_s} B_v(p) d[\delta \tau_v(p)] - \int_0^{p_s} \delta B_v(p) d\tau^0_v(p) \]

Its Dual Form

\[ \delta R_v = \delta B_v(p_s) \tau_v(p_s) + B^0_v(p_s) \delta \tau_v(p_s) - \int_0^{p_s} \delta B_v(p) d\tau_v(p) - \int_0^{p_s} B^0_v(p) d[\delta \tau_v(p)] \]
\[ \delta R_v \approx \beta^0_v(p_s) \tau^0_v(p_s) \delta T_s \]

\[ - \sum_{i=1}^{N} \int_0^{p_s} \beta^0_v(p) \tau^0_v(p) \delta T(p) d\ln \tau^0_v(p) \]

\[ + \sum_{i=1}^{N} \int_0^{p_s} \beta^0_v(p) \tau^0_v(p) \delta U_i(p) \frac{dT(p)}{dU_i^0(p)} d\ln \tau^0_v(p) \]

The effective temperature profile of the \( i \)th absorbing gas:

\[ \delta T_i(p) \equiv \delta T(p) - \delta U_i(p) \frac{dT(p)}{dU_i^0(p)} \]

The effective temperature profile of the \( i \)th absorbing gas:

\[ \delta T_i(p) \equiv \delta T(p) \frac{dU_i(p)}{dU_i^0(p)} - \delta U_i(p) \frac{dT^0(p)}{dU_i^0(p)} \]

Final Linear Form

\[ \delta R_v \approx \beta^0_v(p_s) \tau^0_v(p_s) \delta T_s - \sum_{i=1}^{N} \int_0^{p_s} \beta^0_v(p) \delta T_i(p) \tau^0_v(p) d\ln \tau^0_v(p) \]
\[ \delta T_i(p) \equiv \delta T(p) - \delta U_i(p) \frac{dT(p)}{dU_i^0(p)} \]

\[ U_i(p) = U_i^0(p) + \frac{dU_i^0(p)}{dT(p)} \left[ T(p) - T_i(p) \right] \]

\[ \delta T_i(p) \equiv \delta T(p) \frac{dU_i(p)}{dU_i^0(p)} - \delta U_i(p) \frac{dT^0(p)}{dU_i^0(p)} \]

\[ U_i(p) = U_i^0(p) + \frac{dU_i^0(p)}{dT^0(p)} \left[ T^0(p) - T_i(p) \right] \]

A special case: \( T(p) = T^0(p) \)

\[ \Rightarrow \]

\[ U_i(p) = U_i^0(p) + \frac{dU_i^0(p)}{dT^0(p)} \left[ T(p) - T_i(p) \right] \]

The general case: \( T(p) \neq T^0(p) \)

\[ \Rightarrow \]

\[ U_i(p) = \frac{1}{T^0(p) - T(p)} \times \int_0^p \left( U_i^0(p) + \frac{dU_i^0(p')}{dT^0(p')} \left[ T^0(p') - T_i(p') \right] \right) \frac{dT^0(p')}{dp'} dp' \]

The retrieval quality of absorbing gas profiles depends on the quality of temperature first guess!
Comparison of linear forms of the radiative transfer equation with analytic Jacobians

Bormin Huang, William L. Smith, Hung-Lung Huang, and Harold M. Woolf

\[
R_v = \varepsilon_v B_v(T_s) \tau_v(p_s) - \int_0^{p_s} B_v[T(p)] \frac{d\tau_v(p)}{dp} dp \\
+ r_{v_s} \tau_v(p_s) \int_0^{p_s} B_v[T(p)] \frac{d\tau_v^*(p)}{dp} dp \\
+ R_v^{\text{sun}} \tau_v^{1 + \sec \theta}(p_s)r_{v_s}^{\text{sun}},
\]

\[
\tau_v(p_j) = \exp \left\{ \sum_{k=1}^{n_f} \left[ \sum_{i=1}^{m_f} a_{\nu_i k}^{\text{fixed}} X_{\nu_i k}^{\text{fixed}} \right. \\
+ \sum_{l_w=1}^{m_w} b_{\nu_i l_w}^{\text{water}} X_{\nu_i l_w}^{\text{water}} \\
+ \left. \sum_{l_o=1}^{m_o} \phi_{\nu_i l_o}^{\text{ozone}} X_{\nu_i l_o}^{\text{ozone}} \right\}
\]

\[
\delta R_v = W_T^0 \delta T_s + \sum_{j=1}^L W_T^0(p_j) \delta T(p_j) \\
+ \sum_{i=1}^N \sum_{j=1}^L W_{q_i}^0(p_j) \delta q_i(p_j) + W_{\varepsilon_v}^0 \delta \varepsilon_v \\
+ W_{r_{v_s}}^0 \delta r_{v_s} + W_{r_{v_s}}^{\text{sun}} \delta r_{v_s}^{\text{sun}},
\]

Fig. 1. Comparison of the relative absolute linearization errors between the linear form with exact analytic Jacobians and the linear form with approximate analytic Jacobians.

<table>
<thead>
<tr>
<th>HIRS Channel Number</th>
<th>LHS of Eqs. (44) and (80)</th>
<th>RHS of Eq. (80)</th>
<th>RHS of Eq. (44)</th>
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<tr>
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<td>0.23274</td>
<td>0.22972</td>
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<td>0.21949</td>
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<td>0.21206</td>
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<td>0.01729</td>
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<td>-0.02717</td>
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<td>0.23753</td>
<td>0.30907</td>
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<td>9</td>
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<td>0.05626</td>
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<td>19</td>
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<td>-0.00474</td>
</tr>
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</table>

*The RHS of Eq. (44) is the linear form with approximate analytic Jacobians, whereas the RHS of Eq. (80) is the linear form with exact analytic Jacobians. Equations (44) and (80) have the same LHS, which is calculated by the forward model and represents the true brightness temperature difference in the two conditions.*
**The exact analytical Jacobians approach vs. the tangent-linear & adjoint approach**

1. **Same numerical precision!**

   Example:

   \[
   V_p(T_d) = C \left(\frac{T_0}{T_d}\right)^A e^{(A+B)\left(1-\frac{T_0}{T_d}\right)}
   \]

   Compute

   \[
   \delta V_p = \frac{dV_p}{dT_d} \delta T_d
   \]

2. **The exact analytical Jacobian approach is several times faster!!**

   Example:

   \[
   J = \frac{1}{C} \left(\frac{T_0}{T_d}\right)^A \exp(-(A+B)\left(1-\frac{T_0}{T_d}\right)) \left(-A\cdot T_d + T_0 \cdot (A+B)\right) \cdot T_d^2
   \]

   **Source:** Paul van Delst

   ```plaintext
   ; THE FORWARD MODEL
   proc calculate Vp_FWD, Td, $ ; Input Vp ; Output
   common Constants
   Ratio = To / Td
   X = C * (Ratio^A)
   Y = EXP((A+B) * (ONE - Ratio))
   Vp = X + Y
   end
   
   ; THE TANGENT-LINEAR MODEL
   proc calculate Vp_TL, Td, $ ; Input
   Td_TL, $ ; Input
   Vp_TL ; Output
   common Constants
   Ratio = To / Td
   X = C * (Ratio^A)
   Y = EXP((A+B) * (ONE - Ratio))
   Ratio_TL = (-ONE * To / Td^2) * Td_TL
   X_TL = C * A * (Ratio^A) * Ratio_TL
   Y_TL = -(A+B) * Y * Ratio_TL
   Vp_TL = (X_TL * Y) + (X * Y_TL)
   end
   
   ; THE ADJOINT MODEL
   proc calculate Vp_AD, Td, $ ; Input
   Td_AD ; Input and Output
   Vp_AD, $ ; Input and Output
   common Constants
   Ratio = To / Td
   X = C * (Ratio^A)
   Y = EXP((A+B) * (ONE - Ratio))
   X_AD = Y * Vp_AD
   Y_AD = X * Vp_AD
   Vp_AD = ZERO
   Ratio_AD = -(A+B) * Y * Y_AD
   Y_AD = ZERO
   Ratio_AD = Ratio_AD + (C * A * (Ratio^A)) * X_AD
   X_AD = ZERO
   Td_AD = Td_AD + (-ONE * To / Td^2) * Ratio_AD
   Ratio_AD = ZERO
   end
   ```
Why Use Exact Analytical Jacobians in Retrieval and Data Assimilation?

• Physical Retrieval (1D-Var):

\[ \text{Cost}(X) = \| R^{obs} - R(X) \| \quad \text{or} \quad \text{Cost}(X) = \| R^{obs} - R(X) \| + \lambda \| X - X^0 \| \]

• NWP Data Assimilation (3D/4D-Var):

\[ \text{Cost}(X,...) = \text{Atmospheric Dynamic Term}(X,...) + \| R^{obs} - R(X) \| \]

~ ideal choice for multisperspstral sounders (e.g. HIRS, GOES, MODIS)
~ an underdetermined problem (with respect to a typical forward model)

\[ \text{Cost}(X,...) = \text{Atmospheric Dynamic Term}(X,...) + \| X^{retrieval} - X \| \]

~ ideal choice for hyperspctral sounders (e.g. AIRS, IASI, GIFTS)
~ an overdetermined problem (with respect to a typical forward model)

• Atmospheric dynamic term is basically governed by the Navier-Stokes equation, which has no analytical Jacobians. Thus, its tangent-linear/adjoint models are needed for data assimilation.

• The exact analytical Jacobians for the widely-used McMillin-Fleming-Eyre-Woolf type of radiative transfer/forward models (e.g. RTTOV, RTIASI) are derivable (Huang et al., Applied Optics, 2002).
Summary

1. The classical derivation of the linear form of the RTE by Smith et al. (1991) is reviewed. Its dual form is derived.

2. The original linear form appears to be a special case of its dual form when the temperature first guess happens to be the true temperature profile.

3. Linear forms with *inexact* analytic Jacobians make retrieval results unreliable!

4. The *exact* analytic Jacobians implementation is an efficient alternative to the tangent-linear/adjoint models for hyperspectral retrieval and data assimilation problems with the widely-used McMillin-Fleming-Eyre-Woolf type of forward models (e.g. RTTOV, RTIASI).
Future Work

The Remote Sensing GENOME (Geometrical Exploration of Nonlinear Optimization in Measurement Environment) Project:

*Unveiling the Radiance Hyperspace for Quantifying Geophysical Retrievals*
The remote sensing GENOME project aims to solve the following long-standing fundamental problems in passive remote sensing:

- Given a sensor specification, its forward model and exact Jacobians, to quantify:
  - the information content of each channel, i.e., the best expected contribution from each channel to the retrieval of temperature and absorbing gases at each pressure level,
  - the information content of a sensor, i.e., the best expected retrieval accuracy that sets the statistical limit for all retrieval algorithms,
  - the error of a fast model in terms of the degradation of the best expected retrieval accuracy, as compared to its LBL counterpart,
  - the impact of sensor noise level on the best expected retrieval accuracy,
  - the “first guess tolerance” — a safety measure beyond which no retrieval algorithm can reach the best expected retrieval, and
  - the “retrieval efficiency” — a robustness measure for any retrieval algorithm, as compared to the best expected retrieval.

Applications:

- **Lossy compression retrieval impact studies:**
  to conclude the retrieval degradation (due to lossy compression) by the best expected retrieval accuracy that sets the limit for all possible retrieval algorithms.

- **Optimal channel selection for retrieval with partial channels:**
  to relieve the computational burden in retrieval and data assimilation with the minimum retrieval degradation for hyperspectral sounders (e.g. AIRS, IASI, GIFTS).

- **Future sensor design & trade-off study for risk reduction:**
  to assess the information content (the best expected retrieval accuracy) of a sensor designed with various spectral ranges, ILS resolutions, and noise levels.

*Different living species have different genomes. So do different sensors!*